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The growth of Bargmann functions and the completeness of sequences of coherent states

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Abstract. The growth of Bargmann functions is intimately connected to the density of the zeros of these functions and to the completeness of sequences of coherent states. Using these ideas we find the least density that a sequence of coherent states must have in order to be overcomplete within the space $H(\rho, \sigma)$ of Bargmann functions of an order not exceeding ρ (and of a type not exceeding σ if of order ρ). These results generalize known results on the completeness of von Neumann lattices. The practical significance of this formalism in the context of quantum optics is also discussed.

1. Introduction

Coherent states form an overcomplete set of states in the Hilbert space. In fact it is well known that they form a 'highly overcomplete' set in the sense that there are much smaller subsets of coherent states which are also overcomplete. For example, it can be proved using the Bargmann analytic representation [1], that if $\{z_N\}$ is a convergent sequence to some point z_0 in the complex plane, then the corresponding coherent states $\{|z_N\rangle\}$ form an overcomplete set. A consequence of this is that coherent states on a line form an overcomplete set, resolutions of the identity in terms of these states have recently been studied [2–4]. Using them we can express an arbitrary state as a line integral of coherent states.

It is therefore clear that knowing that a set of coherent states is overcomplete is not only of theoretical interest; it is also of practical interest in the sense that we are encouraged to search for resolutions of the identity which will make possible the expansion of an arbitrary state in terms of these coherent states. Sometimes is not easy to find a resolution of the identity and weaker concepts are also sufficient (for example the concept of frames in wavelets). But it is clear that a prerequisite for going down that route is the question of completeness.

A well known overcomplete set of coherent states is the von Neumann lattice [5, 6]. This is the set of coherent states $\{|S^{1/2}(M + \mathrm{i}N)\rangle\}$ where M, N are integers and S is the area of the lattice cell. It is well known that this set is overcomplete if $S \leqslant \pi$ and undercomplete when $S > \pi$ [5]. The proof uses the Bargmann analytic representation where the growth of a function is intimately related to the density of its zeros and to the completeness of sequences of coherent states. In this paper we generalize these results. In many cases we wish to work in a smaller space than the full Bargmann space. More specifically we consider the space of functions with growth smaller or equal to a certain amount and find what the

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least density that a sequence of coherent states must have in order to be overcomplete. In the process of solving this problem we interpret many of the deep ideas of the 'theory of entire functions, their growth and their zeros', in the context of coherent states.

In section 2 we introduce the Bargmann functions and explain that their zeros imply the orthogonality of the corresponding states with coherent states. In section 3 we discuss the growth of Bargmann functions and give various examples of physical interest. In sections 4 and 5 we consider the space of all Bargmann functions with growth less than a certain value and prove that a sequence of coherent states with density greater (smaller) than a certain value is overcomplete (undercomplete). We also use Hadamard's theorem to show explicitly how we construct states which are orthogonal to a given sequence of coherent states. In section 6 we consider states with arbitrary growth and study their statistical properties (squeezing, antibunching, etc). We conclude in section 7 with a discussion of our results.

2. Bargmann functions and their zeros

We consider a harmonic oscillator and the coherent states:

$$|z\rangle = D(z)|0\rangle = \exp(-\tfrac{1}{2}|z|^2)\sum_{N=0}^{\infty} z^N (N!)^{-1/2}|N\rangle \tag{1}$$

$$D(z) = \exp[za^\dagger - z^* a] \tag{2}$$

where z is a complex number in the complex plane C and $D(z)$ is the displacement operator. For later purposes we also introduce the displaced number states

$$|N; z\rangle = D(z)|N\rangle = \frac{(a^\dagger - z^*)^N}{(N!)^{1/2}}|z\rangle. \tag{3}$$

For $N = 0$ the states $|0; z\rangle$ are simply the coherent states $|z\rangle$. Let $|f\rangle$ be an arbitrary (normalized) state

$$|f\rangle = \sum_{N=0}^{\infty} f_N|N\rangle \qquad \sum_{N=0}^{\infty}|f_N|^2 = 1. \tag{4}$$

We shall use the notation $|f^*\rangle$ for the state

$$|f^*\rangle = \sum_{N=0}^{\infty} f_N^*|N\rangle \qquad \sum_{N=0}^{\infty}|f_N|^2 = 1. \tag{5}$$

In the Bargmann representation the state $|f\rangle$ is represented by the function

$$f(z) = \exp(\tfrac{1}{2}|z|^2)\langle z^*|f\rangle = \exp(\tfrac{1}{2}|z|^2)\langle f^*|z\rangle = \sum_{N=0}^{\infty} f_N z^N (N!)^{-1/2} \tag{6}$$

which is analytic in the complex plane C. Using the resolution of the identity

$$\int \mathrm{d}\mu\,(z)|z\rangle\langle z| = \mathbf{1} \tag{7}$$

$$\mathrm{d}\mu(z) = \pi^{-1}\,\mathrm{d}^2 z \tag{8}$$

we can easily prove that the scalar product of two states $|f\rangle$, $|g\rangle$ can be expressed as

$$\langle f|g\rangle = \int [f(z)]^* g(z)\exp(-|z|^2)\frac{\mathrm{d}^2 z}{\pi}. \tag{9}$$

The creation and annihilation operators are represented as:

$$a = \partial_z \qquad a^\dagger = z. \tag{10}$$

We next point out that if $f(z)$ is the Bargmann function of a state $|f\rangle$ and A is a zero of $f(z)$ (i.e. $f(A) = 0$) then the coherent state $|A\rangle$ is orthogonal to $|f^*\rangle$. We can show this easily with the equation:

$$f(A) = \exp(\tfrac{1}{2}|A|^2)\langle f^*|A\rangle = 0. \tag{11}$$

A more general result is that if A is a zero of $f(z)$ with multiplicity M, then $|f^*\rangle$ is orthogonal to the first M displaced number eigenstates $\{|N; A\rangle; N = 0, \ldots, (M-1)\}$. This can be seen from the equation:

$$\left[\frac{(\partial_z - A)^N}{(N!)^{1/2}} f(z)\right]_{z=A} = \exp\left(\frac{1}{2}|A|^2\right)\langle f^*|\frac{(a^\dagger - A^*)^N}{(N!)^{1/2}}|A\rangle = 0. \tag{12}$$

3. The growth of Bargmann functions

The growth of an analytic function $f(z)$ is characterized by the order ρ and the type σ [7]. If $M(R)$ is the maximum modulus of $f(z)$ for $|z| = R$, then

$$\rho = \lim_{R\to\infty} \sup \frac{\ln\ln M(R)}{\ln R} \tag{13}$$

$$\sigma = \lim_{R\to\infty} \sup \frac{\ln M(R)}{R^\rho}. \tag{14}$$

We shall denote as $H(\rho, \sigma)$ the space of functions of an order not exceeding ρ; and of a type not exceeding σ if of order ρ. Clearly $H(\rho, \sigma)$ is a subset of $H(\rho', \sigma')$ if $\rho < \rho'$; and also if $\rho = \rho'$ and $\sigma < \sigma'$. The sum or product of two entire functions has an order (type) which is at most the larger of the two orders (types). Using equation (9) and the fact that the Bargmann functions are normalizable, we conclude that the Bargmann space is a subspace of $H(2, \frac{1}{2})$.

We consider various examples. The number eigenstate $|N\rangle$ is represented by the function

$$f(z) = z^N (N!)^{-1/2} \tag{15}$$

which is of order 0. In fact any superposition of a finite number of number eigenstates is represented by a (finite) polynomial, which is of order 0.

The coherent state $|A\rangle$ is represented by the function

$$f(z) = \exp[Az - \tfrac{1}{2}|A|^2] \tag{16}$$

which is of order 1 and type $|A|$.

Squeezed states are defined as

$$|A; r, \theta, \lambda\rangle = S(r, \theta, \lambda)|A\rangle \tag{17}$$

$$S(r, \theta, \lambda) = \exp[-\tfrac{1}{4}r\mathrm{e}^{-\mathrm{i}\theta}(a^\dagger)^2 + \tfrac{1}{4}r\mathrm{e}^{\mathrm{i}\theta}a^2]\exp[\mathrm{i}\lambda a^\dagger a]. \tag{18}$$

Using the relation

$$\langle z^*|A; r, \theta, \lambda\rangle = (1 - |\tau|^2)^{1/4}\exp[\alpha z^2 + \beta z + \gamma - \tfrac{1}{2}|z|^2] \tag{19}$$

$$\tau = -\tanh(\tfrac{1}{2}r)\mathrm{e}^{-\mathrm{i}\theta} \tag{20}$$

$$\alpha = \tfrac{1}{2}\tau \tag{21}$$

$$\beta = a\mathrm{e}^{\mathrm{i}\lambda}(1 - |\tau|^2)^{1/2} \tag{22}$$

$$\gamma = -\tfrac{1}{2}\tau^* A^2\mathrm{e}^{2\mathrm{i}\lambda} - \tfrac{1}{2}|A|^2. \tag{23}$$

We find that the Bargmann function is:

$$f(z) = \exp[\tfrac{1}{2}|z|^2]\langle z^*|A; r, \theta, \lambda\rangle = (1 - |\tau|^2)^{1/4} \exp[\alpha z^2 + \beta z + \gamma] \qquad (24)$$

which is of order 2 and type $\sigma = \frac{1}{2}\tanh(\frac{1}{2}r)$.

We also consider the negative binomial states (which are SU(1, 1) Perelomov coherent states [8] in the harmonic oscillator context) given by [9]

$$|w\rangle = (1 - |w|^2)^k \sum_{N=0}^{\infty} \mathrm{d}(k, N) w^N |N\rangle \qquad |w| < 1 \qquad (25)$$

$$\mathrm{d}(k, N) = \left[\frac{\Gamma(N + 2k)}{\Gamma(N + 1)\Gamma(2k)}\right]^{1/2} \qquad k = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \qquad (26)$$

The Bargmann function for these states is

$$f(z) = \sum_{N=0}^{\infty} c_N z^N \qquad (27)$$

$$c_N = \frac{(1 - |w|^2)^k\, \mathrm{d}(k, N) w^N}{(N!)^{1/2}}. \qquad (28)$$

The order and type of such a function, given in the form of a series, can be found by using a theorem (e.g. theorem 2, in chapter 1 of [7]) that states

$$\rho = \lim_{N\to\infty} \sup \frac{N \ln N}{[-\ln |c_N|]} \qquad (29)$$

$$(\sigma e \rho)^{1/\rho} = \lim_{N\to\infty} \sup[N^{1/\rho}|c_N|^{1/N}]. \qquad (30)$$

Using these equations we find that $\rho = 2$ and $\sigma = \frac{1}{2}|w|^2$.

The above examples are of great practical importance in quantum optics, and they all have order ρ which is an integer. It should be stressed that ρ can take all values between 0 and 2 (in fact it can be greater than 2 but then the function is not normalizable). We show this explicitly by giving an example of a state which has the Bargmann function with a given order ρ and given type σ. It is the state:

$$|\rho, \sigma\rangle = \sum_{N=0}^{\infty} f_N |N\rangle \qquad (31)$$

$$f_N = \mathcal{K} \frac{\mathrm{e}^{\mathrm{i}\theta_N} \sigma^{N/\rho} (N!)^{1/2}}{\Gamma(\frac{N}{\rho} + 1)} \qquad (32)$$

where $\mathcal{K}$ is a normalization constant given by:

$$\mathcal{K} = \left[\sum_{N=0}^{\infty} \frac{\sigma^{2N/\rho} N!}{[\Gamma(\frac{N}{\rho} + 1)]^2}\right]^{-\frac{1}{2}} \qquad (33)$$

and $\{\theta_N\}$ are phases. The normalization constant is finite when $0 \leqslant \rho < 2$; and also when $\rho = 2$ and $\sigma < \frac{1}{2}$. Inserting equation (32) into equation (6) we find the Bargmann function for the states (31) and using equations (29) and (30) we show that it is indeed of order ρ and type σ. In the special case $\theta_N = N\theta$ the Bargmann function is given in terms of the Mittag–Leffler function [10, p 206] as $\mathcal{K}E_{1/\rho}(\mathrm{e}^{\mathrm{i}\theta}\sigma^{1/\rho}z)$. In the special case $\rho = 1$ and $\theta_N = N\theta$ the states (31) reduce to the usual coherent states of equation (1) with $z = \sigma \mathrm{e}^{\mathrm{i}\theta}$. For $\rho = 1$ and general phases $\{\theta_N\}$ we obtain the generalized coherent states studied in [11].

4. Completeness of a sequence of coherent states in $H(\rho, \sigma)$

Let $A_1, \ldots, A_N, \ldots$ be a sequence of complex numbers such that

$$0 < |A_1| \leqslant |A_2| \leqslant |A_3| \leqslant \cdots \tag{34}$$

$$\lim_{N\to\infty} |A_N| = \infty. \tag{35}$$

The convergence exponent ρ_1 of this sequence is the infimum of positive numbers λ for which

$$\sum_{N=1}^{\infty} |A_N|^{-\lambda} < \infty \tag{36}$$

i.e. the series converges. Let $n(R)$ be the number of terms of this sequence enclosed within the circle $|z| < R$. It is known [7] that an alternative equivalent definition of the convergence exponent ρ_1 is:

$$\rho_1 = \lim_{R\to\infty} \sup \frac{\ln n(R)}{\ln R}. \tag{37}$$

In other words, the number of terms of the sequence within a very large circle of radius R, is proportional to R^{ρ_1}. For a given convergence exponent ρ_1 a more refined description of the density of the set $\{A_N\}$ is done with the numbers

$$\Delta = \lim_{R\to\infty} \sup \frac{n(R)}{R^{\rho_1}} \tag{38}$$

$$\delta = \lim_{R\to\infty} \inf \frac{n(R)}{R^{\rho_1}} \tag{39}$$

which we call upper and lower density, correspondingly. These two quantities are different in 'oscillatory' cases. When the (ordinary) limit exists, $\delta = \Delta$. The quantities ρ_1, Δ, δ characterize the density of the sequence $\{A_N\}$.

We now consider the set of coherent states $\{|A_N\rangle\}$ and examine its completeness with respect to functions in the space $H(\rho, \sigma)$. We shall say that the density of this sequence of coherent states is (ρ_1, δ) if the sequence of complex numbers $\{A_N\}$ has convergent exponent (ρ_1) and lower density δ. We shall also say that the density of this sequence of coherent states is greater (smaller) than (ρ_1, δ) if the sequence of complex numbers $\{A_N\}$ has a convergent exponent greater (smaller) than ρ_1; or if it has a convergent exponent equal to ρ_1 but has a lower density greater (smaller) than δ.

Incompleteness of a set of coherent states $\{|A_N\rangle\}$ with respect to the space $H(\rho, \sigma)$ is intimately connected to the existence of a Bargmann function $f(z)$ in $H(\rho, \sigma)$ with zeros at all $\{A_N\}$. This is because the existence of such a function implies the existence of a state that is orthogonal to all the coherent states $\{|A_N\rangle\}$.

It is known that the convergent exponent of the zeros of an entire function does not exceed its order [7]. Therefore, for functions in $H(\rho, \sigma)$ (with any σ) any sequence of coherent states $\{|A_N\rangle\}$ with a convergence exponent greater than ρ, is at least complete. In fact it is overcomplete because the same conclusion can be reached even if we leave out the first K coherent states (where K is any finite number).

We next consider a set of coherent states with convergent exponent equal to ρ and upper and lower densities Δ and δ, correspondingly. Using theorem (9.1.1) in Boas [7] we conclude that this set is also overcomplete if either of the following relations is satisfied:

$$\sigma\rho \leqslant \delta \tag{40}$$

$$e\sigma\rho \leqslant \Delta. \tag{41}$$

We therefore conclude that for functions in $H(\rho, \sigma)$ any sequence of coherent states with a density greater than $(\rho, \sigma\rho)$ is overcomplete. In the next section we will show that when the density is smaller than $(\rho, \sigma\rho)$ the corresponding set of coherent states is undercomplete.

5. Hadamard's theorem and its physical interpretation in the context of coherent states

Hadamard's theorem states that an entire function of order ρ can be factorized as

$$f(z) = P(z)\exp[Q_q(z)] \tag{42}$$

where

$$P(z) = z^m \prod_{N=1}^{\infty} E(A_N, p) \tag{43}$$

$$E(A_N, 0) = 1 - \frac{z}{A_N} \tag{44}$$

$$E(A_N, p) = \left(1 - \frac{z}{A_N}\right)\exp\left[\frac{z}{A_N} + \frac{z^2}{2A_N^2} + \cdots + \frac{z^p}{pA_N^p}\right] \qquad p \geqslant 1 \tag{45}$$

$Q_q(z)$ is a polynomial of degree q and p is an integer. The maximum of (p, q) is called genus of $f(z)$ and does not exceed the order ρ. The $\{A_N\}$ are clearly zeros of this function and it is known [7] that their convergent exponent does not exceed the order ρ; and if equal to ρ then the lower and upper densities satisfy the relations:

$$\sigma\rho \geqslant \delta \tag{46}$$

$$e\sigma\rho \geqslant \Delta. \tag{47}$$

For Bargmann functions $\rho \leqslant 2$ and therefore p and q can only take the values $0, 1, 2$. In order to interpret this theorem in the context of coherent states we first point out that when q takes the values $0, 1, 2$ the $\exp[Q_q(z)]$ represents (up to a normalization constant) the vacuum state, a coherent state and a squeezed state (see equations (16) and (19)). Using the notation $|Q_q\rangle$ for the state represented by the Bargmann function $\exp[Q_q(z)]$ we have

$$|Q_0\rangle = |0\rangle \tag{48}$$

$$|Q_1\rangle = |A\rangle \tag{49}$$

$$|Q_2\rangle = |A; r, \theta, \lambda\rangle. \tag{50}$$

Using equation (10) we express the E-factors in terms of creation operators:

$$\hat{E}(A_N, 0) = 1 - \frac{a^\dagger}{A_N} \tag{51}$$

$$\hat{E}(A_N, 1) = \left[1 - \frac{a^\dagger}{A_N}\right]\exp\left[\frac{a^\dagger}{A_N}\right] \tag{52}$$

$$\hat{E}(A_N, 2) = \left[1 - \frac{a^\dagger}{A_N}\right]\exp\left[\frac{a^\dagger}{A_N} + \frac{(a^\dagger)^2}{2A_N^2}\right] \tag{53}$$

and the z^m as the operator $(a^\dagger)^m$. Note that all these operators commute with each other. Using Hadamard's theorem we can now prove that any state can be expressed as

$$|f\rangle = (a^\dagger)^m \prod_{N=1}^{\infty} \hat{E}(A_N, p)|Q_q\rangle. \tag{54}$$

Then q, p take the values $0, 1, 2$ for states in $H(2, \frac{1}{2})$. If we are interested in a subspace $H(\rho, \sigma)$ then if $1 \leqslant \rho < 2$ the q, p take the values 0, 1; and if $0 \leqslant \rho < 1$ then $q = p = 0$.

The state $|f^*\rangle$ is perpendicular to all coherent states $\{|A_N\rangle\}$ whose density, as we explained, is smaller than $(\rho, \sigma\rho)$. This proves the incompleteness of this set of coherent states in $H(\rho, \sigma)$. We therefore conclude that *for functions in $H(\rho, \sigma)$ any sequence of coherent states with a density greater (smaller) than $(\rho, \sigma\rho)$ is overcomplete (undercomplete)*.

In the special case of the full Bargmann space $\rho = 2$ and $\sigma = \frac{1}{2}$ and therefore the $(\rho, \sigma\rho)$ is $(2, 1)$. The von Neumann lattice of coherent states $\{|S^{1/2}(M + \mathrm{i}N)\rangle\}$ where M, N are integers and S is the area of the lattice cell, has convergence exponent $\rho_1 = 2$ and lower and upper density $\delta = \Delta = S/\pi$. It is clear that in this special case our general results reduce to the well known results on von Neumann lattices.

We would also like to point out that a sequence of coherent states with density equal to $(\rho, \sigma\rho)$ can be overcomplete or undercomplete. An example is the von Neumann lattice with $S = \pi$. If we consider the full lattice it is known to be overcomplete by one state [5]. However, we can consider the same lattice with a finite number of coherent states added to it; or a finite number of coherent states subtracted from it. The fact that we add or subtract a *finite* number of coherent states does not change its density (ρ_1, δ). This 'enlarged' or 'truncated' von Neumann lattice is now overcomplete or undercomplete by a finite number of states.

6. Quantum statistical properties of states with growth (ρ, σ)

There has been a lot of interest in literature on quantum optics on states that exhibit squeezing and antibunching. For this reason we study here the statistical and uncertainty properties of the states (31). We calculate the average number of photons

$$\langle N\rangle = \sum_{N=0}^{\infty} N|f_N|^2 \tag{55}$$

the second-order correlation

$$g^{(2)} = \frac{\langle (a^\dagger)^2 a^2\rangle}{\langle a^\dagger a\rangle^2} = \frac{\langle N^2\rangle - \langle N\rangle}{\langle N\rangle^2} \tag{56}$$

the uncertainty

$$\Delta x = [\langle x^2\rangle - \langle x\rangle^2]^{1/2} \tag{57}$$

$$\langle x^i\rangle = \sum_{N,M} f_N f_M^* \langle N| \left(\frac{a + a^\dagger}{2^{1/2}}\right)^i |M\rangle \tag{58}$$

and the uncertainty product $\Delta x \Delta p$. For simplicity we consider the case $\theta_N = 0$. The infinite sums have been truncated at $N = 32$; and the region of ρ has been limited to $0.2 < \rho < 2$. Larger values of N or smaller values of ρ require the evaluation of the gamma function at large values and this leads to numerical difficulties. We have checked that the average number of photons $\langle N\rangle$ in all the cases that we have considered is much less than 32 so this is a good approximation. We should also point out that larger values of σ than those considered, lead to average numbers of photons close to 32 (especially for ρ near 2) and therefore to inaccurate numerical results.

Numerical results are presented in figures 1–4. It is seen that for ρ less than 1.6 the uncertainty product $\Delta x \Delta p$ takes values very close to $\frac{1}{2}$ (a typical value is 0.5001). The

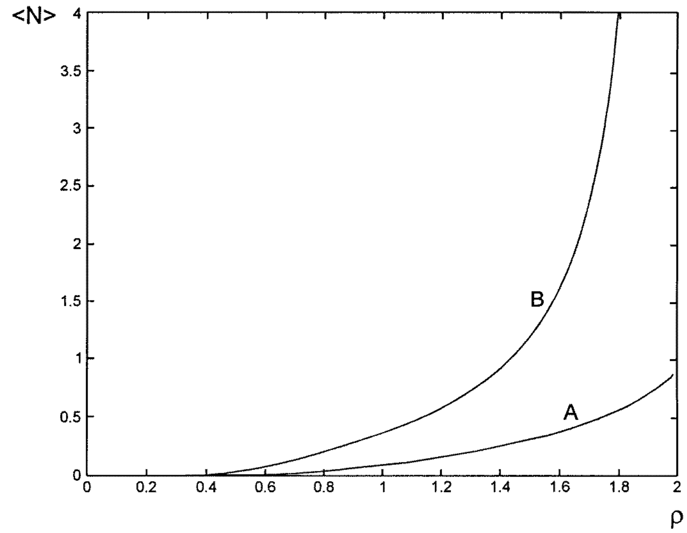

Figure 1. The average number of photons $\langle N \rangle$ for the states (31), as a function of ρ for (A) $\sigma = 0.3$; and (B) $\sigma = 0.6$.

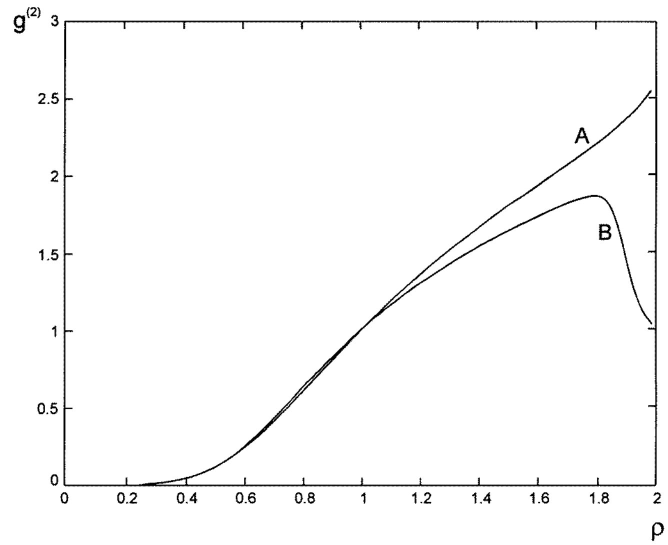

Figure 2. $g^{(2)}$ for the states (31), as a function of ρ for (A) $\sigma = 0.3$; and (B) $\sigma = 0.6$.

uncertainty Δx shows some modest squeezing (i.e. $\Delta x < 2^{-1/2}$) around $\rho = 0.7$. The $g^{(2)}$ shows strong antibunching (i.e. $g^{(2)} < 1$) for ρ less than 1.

7. Discussion

The theory of entire functions and their zeros, has important implications on the theory of coherent states; on the related area of frames and wavelets [12]; and also on other areas of physics (e.g. [13]). In this paper we have shown how the growth of a Bargmann function

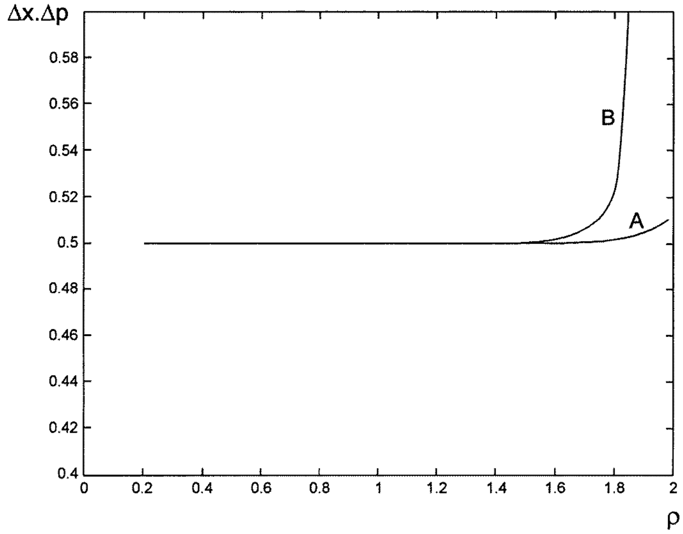

Figure 3. $\Delta x \Delta p$ for the states (31), as a function of ρ for (A) $\sigma = 0.3$; and (B) $\sigma = 0.6$.

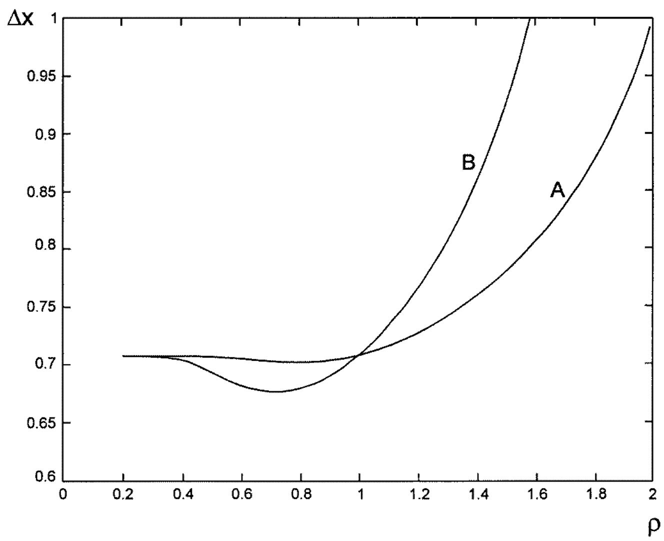

Figure 4. Δx for the states (31), as a function of ρ for (A) $\sigma = 0.3$; and (B) $\sigma = 0.6$.

$f(z)$ is related to its zeros; and to coherent states which are orthogonal to the corresponding state $|f^*\rangle$. Using Hadamard's theorem we have constructed explicitly in equation (54) a state orthogonal to a given sequence of coherent states. We have also shown that with respect to the space $H(\rho, \sigma)$ a sequence of coherent states with density greater (smaller) than $(\rho, \sigma\rho)$ is overcomplete (undercomplete).

On the more applied side we have examined the growth of various states of practical interest (coherent states, squeezed states, etc). We have also considered the states (31) of growth (ρ, σ) and studied their statistical properties (squeezing, antibunching, etc).

Another potential application of the ideas of this paper is in the area of Gabor transforms in signal processing. Gabor [14] and Ville [15] initiated the so-called time-frequency methods in signal processing (the analogue of phase space methods in quantum mechanics). He used Gaussian signals (the analogue of coherent states) on a von Neumann lattice as a basis in the space of signals (for more recent work see [16] and references therein). In this line of research, on many occasions we might be interested only in signals within a subspace such as $H(\rho, \sigma)$. The work of this paper gives the least density of a basis of Gaussians for these signals. Also, when we use a basis of Gaussians which is less dense than a von Neumann lattice, we want to know what type of signals this basis is not able to detect (i.e. which signals are orthogonal to the basis). Hadamard's theorem answers this question precisely.

In conclusion, we believe that the results of this paper provide a bridge between the theory of growth and zeros of entire functions, and the theory of coherent states.

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